

# New series representations for zeta numbers using polylogarithmic identities in combination with a polynomial description of Bernoulli numbers

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## Abstract

With this paper we introduce a new series representation of  $\zeta(3)$ , which is based on the Clausen representation of odd integer zeta values. Although, relatively fast converging series based on the Clausen representation exist for  $\zeta(3)$ , their convergence behavior is very slow compared to BBP-type formulas, and as a consequence they are not used for explicit numerical computations. The reason is found in the fact that the corresponding Clausen function can be calculated analytically for a few rational arguments only, where  $x = \frac{1}{6}$  is the smallest one. Using polylogarithmic identities in combination with a polynomial description of the even Bernoulli numbers, the convergence behavior of the Clausen-type representation has been improved to a level that allows us to challenge ultimately all BBP-type formulas available for  $\zeta(3)$ . We present an explicit numerical comparison between one of the best available BBP formulas and our formalism. Furthermore, we demonstrate by an explicit computation using the first four terms in our series representation only that  $\zeta(3)$  results with an accuracy of  $2 \cdot 10^{-26}$ , where our computation guarantees on each approximation level for an analytical expression for  $\zeta(3)$ .

## 1 INTRODUCTION

During the last decades BBP-type formulas have been established as the technique of choice for very fast digit extraction of mathematical constants, as for example,  $\pi$ ,  $\ln(2)$ ,  $\zeta(3)$  or  $\zeta(5)$  [1, 2, 3, 4, 5, 6, 7]. This is because the corresponding algorithms are simply to implement, where the need of computer memory is very low and no multiple precision arithmetic software is

required [6]. Apart from digit extraction interest has grown in BBP-type formulas in context with statistical randomness of the digit expansions of polylogarithmic constants [1].

However, a shortcoming of BBP-type formulas is that a variety of binary degree-1 and degree-2 formulas exist, but only a few ternary (base 3) or even higher degree BBP-type formulas for polylogarithmic constants are known. The reason is found in the strong increase of complexity of polylogarithmic functional equations as a function of the corresponding binary degree. An example is given by Adegoke [8] for a polylogarithmic functional equation of degree 5, where no BBP-type formula for  $\zeta(7)$  or higher odd-integer zeta values has been discovered so far [8].

Concerning the computation of odd-integer valued zeta numbers the so called Clausen representation of zeta numbers [9, 10, 11] allows for relatively fast digit extraction, which is not restricted to  $\zeta(3)$  and  $\zeta(5)$ . We have combined this approach recently with a new polynomial representation of the Bernoulli numbers in connection with Bendersky's L-numbers [12], which appear in context with the logarithmic Gamma function [13]. As a first application approximate calculations of  $\zeta(3)$ ,  $\zeta(5)$  and  $\zeta(7)$  in terms this polynomial representation had been presented, where this computational procedure is applicable to all  $\zeta$ -values with integer arguments, as well as to related numbers like Catalan's constant. Compared to digit extraction via corresponding BBP-type formulas the convergence behavior is not really competitive because the Clausen functions can be calculated analytically for a few rational arguments only, where  $x = \frac{1}{6}$  is the smallest one. In principle one may argue that the speed up in the convergence should be significant if one would be able to find smaller real-type arguments which also allows one for an analytical computation of the corresponding Clausen function. This is indeed possible by the use of polylogarithmic ladder identities, which exist for  $\zeta(3)$  and  $\zeta(5)$ . In this contribution we present a first application to the numerical computation of  $\zeta(3)$  where we combine polylogarithmic identities with our polynomial description of Bernoulli numbers to challenge one of the best available BBP-type formulas typically used for digit extraction of  $\zeta(3)$  [2]. We demonstrate by an explicit computation that a fast computation of  $\zeta(3)$  is possible, for example with an accuracy of about  $10^{-26}$ . Furthermore, we demonstrate that our approach guarantees for an analytical expression of  $\zeta(3)$  independently from the requested numerical accuracy. At last we present an explicit numerical computation which shows that our series representation of  $\zeta(3)$  converges more than six orders of magnitude faster compared to the famous BBP-type formula discovered first by Bailey and coworkers [14].

The paper is organized as follows: in section 2 we remark on the Clausen representation of odd-integer zeta numbers and present a first computation of  $\zeta(3)$  by use of a well known polylogarithmic ladder identity for  $Li_3\left(\frac{1}{2}\right)$ . In section 3 we introduce our polynomial representation of the even Bernoulli numbers and as a consequence for  $\zeta(2n)$ ,  $n \in \mathbb{N}$ . This approach is then combined with the Clausen representation of  $\zeta(3)$  to achieve a fast converging series representation, which guarantees on each approximation level for an analytical expression of  $\zeta(3)$ . Furthermore, we demonstrate that the iterated use of an appropriate polylogarithmic functional equation for  $Li_3(x)$  allows for a tremendous speed up of the convergence behavior of our series representation. In section 4 we summarize our results.

## 2 Clausen representation of zeta numbers

Well-known for a long time is the famous Euler representation [9, 10] of  $\zeta(2n)$  with  $n \in \mathbb{N}$ :

$$\zeta(2n) = (-1)^{n+1} B_{2n} \frac{(2\pi)^{2n}}{2(2n)!}. \quad (2.1)$$

For odd integer numbers, as for example, for  $n = 3$  one finds [11, 15]:

### Lemma 2.1

$$\begin{aligned} Cl_3(x) = \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^3} = \zeta(3) - 3\pi^2 x^2 + 2\pi^2 x^2 \ln(2\pi|x|) \\ - 8\pi^2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n(2n+1)(2n+2)} x^{2n+2}, \end{aligned} \quad (2.2)$$

with  $x \in \mathbb{R}$ . A computation of the Clausen function  $Cl_3(x)$  for the argument  $x = \frac{1}{6}$  results in [11]:

$$\zeta(3) = \frac{\pi^2}{8} - \frac{\pi^2}{12} \ln\left(\frac{\pi}{3}\right) + \frac{\pi^2}{3} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n(2n+1)(2n+2)} \left(\frac{1}{6}\right)^{2n}. \quad (2.3)$$

Equation (2.3) converges rather fast, and obviously the convergence could be improved using smaller arguments for the Clausen function. Unfortunately, this is not for possible  $x \in \mathbb{Q}$ , as for smaller rational arguments as  $x = \frac{1}{6}$  partial sums remain in the computation of the Clausen function, which are not expressible in terms of  $\zeta(3)$ . The way out is the use of polylogarithmic functions, which are widely used in so called BBP formulas [3, 8, 16]. For example, it follows for  $Li_3\left(\frac{1}{2}\right)$  [3]:

$$Li_3\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\frac{1}{2}\right)^n = \frac{7}{8}\zeta(3) + \frac{1}{6}(\ln(2))^3 - \frac{\pi^2}{12}\ln(2). \quad (2.4)$$

Reformulating  $Li_3\left(\frac{1}{2}\right)$  in the following way:

### Lemma 2.2

$$Li_3\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^3} e^{in\theta}, \quad (2.5)$$

with  $\theta = i \ln(2)$  the computation in terms of the corresponding Clausen functions  $Cl_3(x)$  and  $Sl_3(x)$  results in:

### Lemma 3.2

$$\begin{aligned} \zeta(3) = & \frac{2\pi^2}{3} \ln(2) - 6(\ln(2))^2 + \frac{2}{3}(\ln(2))^3 + 4(\ln(2))^2 \ln(\ln(2)) \\ & + 16(\ln(2))^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \zeta(2n)}{2n(2n+1)(2n+2)} \left(\frac{\ln(2)}{2\pi}\right)^{2n}, \end{aligned} \quad (2.6)$$

where  $Sl_3(x)$  is known analytically. Using the corresponding expression for  $Li_3(\frac{1}{2})$  the new argument  $x = \frac{\ln(2)}{2\pi}$  is  $x \approx \frac{1}{9}$  instead of  $x = \frac{1}{6}$ , and as a consequence the convergence is much faster. Furthermore, this series shows up with an alternating sign which provides some benefit in estimating the convergence properties. For example,  $\zeta(3)$  results from the sum of the explicit terms including the first term ( $n=1$ ) from infinite series with an error of  $\delta \approx 10^{-07}$ .

One may notice that the argument  $x = \frac{1}{6}$  works for all Clausen functions  $Cl_{2n+1}$  as the well known identity exists [11]:

$$Cl_{2n+1}\left(\frac{\pi}{3}\right) = \frac{1}{2} (1 - 2^{-2n}) (1 - 3^{-2n}) \zeta(2n + 1) . \quad (2.7)$$

This procedure is also applicable to  $\zeta(2)$ , with [3, 16]:

$$Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2} (\ln(2))^2 . \quad (2.8)$$

It follows then for  $\zeta(2)$ :

$$\zeta(2) = 2\ln(2)(1 - \ln(\ln(2))) - \frac{1}{2} (\ln(2))^2 - 4\ln(2) \sum_{n=1}^{\infty} \frac{(-)^{n+1} \zeta(2n)}{2n(2n+1)} \left(\frac{\ln(2)}{2\pi}\right)^{2n} . \quad (2.9)$$

Unfortunately, for zeta values with larger integer argument a similar computation seems not possible because for  $Li_n(\frac{1}{2})$  with  $n > 3$  no closed expressions are known [17, 18, 19]. For example, it follows for  $Li_4(\frac{1}{2})$ :

$$Li_4\left(\frac{1}{2}\right) = \frac{15}{16} \zeta(4) - \frac{7}{8} \zeta(3) \ln(2) + \frac{1}{4} \zeta(2) (\ln(2))^2 + \sum_{n=1}^{\infty} \frac{(-)^n}{(n+1)^3} H_n , \quad (2.10)$$

where  $H_n$  denotes the ordinary finite harmonic series. For the corresponding infinite series no analytical expression exists. As a consequence the computational scheme introduced here is applicable to a non-trivial computation of  $\zeta(3)$  only, as  $\zeta(2)$  is known from Eq. (2.1) explicitly.

### 3 Explicit calculation of $\zeta(3)$ in terms of polylogarithmic identities

To further improve the convergence in the calculation of  $\zeta(3)$  a polynomial representation of the Bernoulli numbers will be used [13]:

#### Proposition 2.1

$$\zeta(2n) = \frac{\zeta(2)^n}{(2n-1)} \sum_{l=1}^n (-)^{l+1} \binom{n+2-l}{2} P^{(l)}(n), \quad (3.1)$$

where the P-polynomials are available from the following recursion relation [13]:

$$P^{(n-l+1)}(n) = 6^n \frac{l-1}{2n-l} \sum_{i=l-1}^{n-1} \frac{P^{i-l+2}(i)}{6^i (2n-2i)} , l > 1 . \quad (3.2)$$

with  $P^{(1)}(n) = \frac{1}{n}$ . As an example, the next three Polynomials result to:

$$P^{(2)}(n) = \frac{3}{2 * 5} \quad (3.3)$$

$$P^{(3)}(n) = \frac{3(21n - 43)}{2^3 * 5^2 * 7} \quad (3.4)$$

$$P^{(4)}(n) = \frac{63n^2 - 387n + 590}{2^4 * 5^3 * 7} \quad (3.5)$$

With this we have:

$$\begin{aligned} \ln \sin(\pi x) &= \ln(\pi x) \\ &- \sum_{n=1}^{\infty} \frac{2\zeta(2)^n}{(2n-1)2n} \left( \sum_{l=1}^n (-)^{l+1} \binom{n+2-l}{2} P^{(l)}(n) \right) x^{2n}. \end{aligned} \quad (3.6)$$

For  $\zeta(3)$  it follows then:

$$\begin{aligned} \zeta(3) &= \frac{2\pi^2}{3} \ln(2) - 6(\ln(2))^2 + \frac{2}{3}(\ln(2))^3 + 4(\ln(2))^2 \ln(\ln(2)) \\ &+ 192 \sum_{i=1}^{\infty} c_i \left( \frac{\ln(2)}{2\sqrt{6}} \right)^{2i}, \end{aligned} \quad (3.7)$$

with

$$c_i = \sum_{n=1}^{\infty} \frac{(-)^{n+1} n(n+1) P^{(i)}(n+i-1)}{(2n+2i-3)(2n+2i-2)(2n+2i-1)(2n+2i)} \left( \frac{\ln(2)}{2\sqrt{6}} \right)^{2n}. \quad (3.8)$$

Using furthermore the polylogarithmic identities [7]:

$$\begin{aligned} Li_3\left(\frac{3}{4}\right) + 2Li_3\left(\frac{1}{3}\right) + Li_3\left(\frac{1}{4}\right) &= \frac{19}{6}\zeta(3) + \frac{1}{3}(\ln(3))^3 - \frac{4}{3}(\ln(2))^3 \\ &- \frac{\pi^2}{3} \ln(2) + 2\ln\left(\frac{4}{3}\right) (\ln(2))^2 \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} Li_3\left(\frac{1}{3}\right) + \frac{1}{4}Li_3\left(\frac{1}{4}\right) + Li_3\left(\frac{2}{3}\right) &= \frac{15}{8}\zeta(3) + \frac{1}{6}(\ln(2))^3 - \frac{\pi^2}{12} \ln(2) \\ &- \frac{1}{6} \left( \ln\left(\frac{3}{2}\right) \right)^3 + \frac{1}{2} \ln(3) \left( \ln\left(\frac{3}{2}\right) \right)^2 - \frac{\pi^2}{6} \ln\left(\frac{3}{2}\right), \end{aligned} \quad (3.10)$$

together with the following functional equation for  $Li_3(x)$ : [7]

$$\begin{aligned} \frac{7}{4}\zeta(3) &= \frac{1}{4}Li_3\left(\left(\frac{1-x}{1+x}\right)^2\right) - 2Li_3\left(\frac{1-x}{1+x}\right) + 2Li_3(1-x) \\ &+ Li_3\left(\frac{1}{1+x}\right) - \frac{1}{2}Li_3(1-x^2) + \frac{\pi^2}{6} \ln(1+x) - \frac{1}{3}(\ln(1+x))^3, \end{aligned} \quad (3.11)$$

a new identity results with all three arguments of  $Li_3(x)$  much closer to 1:

$$\begin{aligned}
6Li_3\left(\frac{2}{3}\right) + 3Li_3\left(\frac{3}{4}\right) - Li_3\left(\frac{8}{9}\right) &= \frac{91}{12}\zeta(3) - \frac{\pi^2}{2}\ln(2) + \frac{7}{3}(\ln(2))^3 - \frac{1}{3}(\ln(3))^3 \\
&- \frac{1}{3}\left(\ln\left(\frac{3}{2}\right)\right)^3 - 2\ln\left(\frac{4}{3}\right)(\ln(2))^2 + \ln(3)\left(\ln\left(\frac{3}{2}\right)\right)^2 \\
&+ \frac{2}{3}\left(\ln\left(\frac{4}{3}\right)\right)^3.
\end{aligned} \tag{3.12}$$

In a next step we compute  $Li_3(x)$  in terms of the corresponding Clausen function by use of the polynomial representation of the Bernoulli numbers and with the help of Eq. (3.1). It follows then for the polylogarithmic function  $Li_3(x)$ :

$$\begin{aligned}
Li_3(x) &= \zeta(3) - \frac{\pi^2}{6}\ln\left(\frac{1}{x}\right) + \frac{1}{12}\left(\ln\left(\frac{1}{x}\right)\right)^3 + \frac{3}{4}\left(\ln\left(\frac{1}{x}\right)\right)^2 - \frac{1}{2}\left(\ln\left(\frac{1}{x}\right)\right)^2 \ln\left(\ln\left(\frac{1}{x}\right)\right) \\
&- 24\left(\ln\left(\frac{1}{x}\right)\right)^2 \sum_{n=1}^{\infty} \frac{2(-)^{n+1} \sum_{l=1}^n (-)^{l+1} \binom{n+2-l}{2} P^{(l)}(n)}{(2n-1)2n(2n+1)(2n+2)} \left(\frac{\ln\left(\frac{1}{x}\right)}{2\sqrt{6}}\right)^{2n+2}.
\end{aligned} \tag{3.13}$$

For  $\zeta(3)$  this gives:

$$\begin{aligned}
\zeta(3) &= \frac{2\pi^2}{5}\ln(3) - \frac{54}{5}\left(\ln\left(\frac{3}{2}\right)\right)^2 - \frac{27}{5}\left(\ln\left(\frac{4}{3}\right)\right)^2 + \frac{9}{5}\left(\ln\left(\frac{9}{8}\right)\right)^2 - \frac{6}{5}\left(\ln\left(\frac{3}{2}\right)\right)^3 \\
&+ \left(\ln\left(\frac{4}{3}\right)\right)^3 + \frac{1}{5}\left(\ln\left(\frac{9}{8}\right)\right)^3 + \frac{28}{5}(\ln(2))^3 - \frac{4}{5}(\ln(3))^3 - \frac{4}{5}\left(\ln\left(\frac{3}{2}\right)\right)^3 \\
&+ \frac{36}{5}\left(\ln\left(\frac{3}{2}\right)\right)^2 \ln\left(\ln\left(\frac{3}{2}\right)\right) + \frac{18}{5}\left(\ln\left(\frac{4}{3}\right)\right)^2 \ln\left(\ln\left(\frac{4}{3}\right)\right) \\
&- \frac{6}{5}\left(\ln\left(\frac{9}{8}\right)\right)^2 \ln\left(\ln\left(\frac{9}{8}\right)\right) \\
&+ \frac{1728}{5} \sum_{i=1}^{\infty} a_i \left(\frac{\ln(\frac{3}{2})}{2\sqrt{6}}\right)^{2i} + \frac{864}{5} \sum_{i=1}^{\infty} b_i \left(\frac{\ln(\frac{4}{3})}{2\sqrt{6}}\right)^{2i} - \frac{288}{5} \sum_{i=1}^{\infty} c_i \left(\frac{\ln(\frac{9}{8})}{2\sqrt{6}}\right)^{2i},
\end{aligned} \tag{3.14}$$

with

$$a_i = \sum_{n=1}^{\infty} \frac{(-)^{n+1} n(n+1) P^{(i)}(n)}{(2n+2i-3)(2n+2i-2)(2n+2i-1)(2n+2i)} \left(\frac{\ln(\frac{3}{2})}{2\sqrt{6}}\right)^{2n}, \tag{3.15}$$

$$b_i = \sum_{n=1}^{\infty} \frac{(-)^{n+1} n(n+1) P^{(i)}(n)}{(2n+2i-3)(2n+2i-2)(2n+2i-1)(2n+2i)} \left(\frac{\ln(\frac{4}{3})}{2\sqrt{6}}\right)^{2n}, \tag{3.16}$$

and

$$c_i = \sum_{n=1}^{\infty} \frac{(-)^{n+1} n(n+1) P^{(i)}(n)}{(2n+2i-3)(2n+2i-2)(2n+2i-1)(2n+2i)} \left( \frac{\ln(\frac{9}{8})}{2\sqrt{6}} \right)^{2n}, \quad (3.17)$$

with the arguments  $\frac{\ln(\frac{3}{2})}{2\sqrt{6}}$ ,  $\frac{\ln(\frac{4}{3})}{2\sqrt{6}}$  and  $\frac{\ln(\frac{9}{8})}{2\sqrt{6}}$  for the coefficients  $a_i$ ,  $b_i$  and  $c_i$ . Summing up the first four terms ( $i=1,2,3,4$ ) from each of the three infinite series together with the explicit terms for an approximate computation  $\zeta(3)$  follows with an error of  $\delta \approx 0.3 * 10^{-17}$ . This is only two orders of magnitude slower in the convergence when compared, for example, to the famous BBP formula [14]:

$$\begin{aligned} \zeta(3) = \frac{1}{672} \sum_{k=0}^{\infty} \left( \frac{1}{4096} \right)^k & \left[ \begin{aligned} & \frac{2048}{(24k+1)^3} - \frac{11264}{(24k+2)^3} - \frac{1024}{(24k+3)^3} + \frac{11776}{(24k+4)^3} \\ & - \frac{512}{(24k+5)^3} + \frac{4096}{(24k+6)^3} + \frac{256}{(24k+7)^3} + \frac{3456}{(24k+8)^3} \\ & + \frac{128}{(24k+9)^3} - \frac{704}{(24k+10)^3} - \frac{64}{(24k+11)^3} - \frac{128}{(24k+12)^3} \\ & - \frac{32}{(24k+13)^3} - \frac{176}{(24k+14)^3} + \frac{16}{(24k+15)^3} + \frac{216}{(24k+16)^3} \\ & + \frac{8}{(24k+17)^3} + \frac{64}{(24k+18)^3} - \frac{4}{(24k+19)^3} + \frac{46}{(24k+20)^3} \\ & - \frac{2}{(24k+21)^3} - \frac{11}{(24k+22)^3} + \frac{1}{(24k+18)^3} \end{aligned} \right]. \quad (3.18) \end{aligned}$$

Within an iterated use of the functional equation (3.10) better and better approximations can be found. The slowest convergence is found now by  $Li_3(\frac{2}{3})$ . As a consequence we rewrite with the help of (3.10)  $Li_3(\frac{2}{3})$ . It follows first for the polylogarithmic function:

$$\begin{aligned} Li_3\left(\frac{2}{3}\right) &= 8Li_3\left(\sqrt{\frac{2}{3}}\right) - 8Li_3\left(\frac{2\sqrt{2}}{\sqrt{2}+\sqrt{3}}\right) - 8Li_3\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{3}}\right) \\ &+ 2Li_3\left(\frac{4\sqrt{6}}{5+2\sqrt{6}}\right) + 7\zeta(3) - \frac{2\pi^2}{3} \ln\left(\frac{2\sqrt{2}}{\sqrt{2}+\sqrt{3}}\right) \\ &+ \frac{4}{3} \left[ \ln\left(\frac{2\sqrt{2}}{\sqrt{2}+\sqrt{3}}\right) \right]^3, \quad (3.19) \end{aligned}$$

and finally  $Li_3\left(\frac{2}{3}\right)$  results to:

$$\begin{aligned}
Li_3\left(\frac{2}{3}\right) &= \zeta(3) - \frac{2\pi^2}{3} \ln\left(\frac{3}{2}\right) + \frac{1}{12} \left(\ln\left(\frac{3}{2}\right)\right)^3 + \frac{3}{2} \left(\ln\left(\frac{3}{2}\right)\right)^2 \\
&- \left(\ln\left(\frac{3}{2}\right)\right)^2 \ln\left(\frac{1}{2} \ln\left(\frac{3}{2}\right)\right) + \frac{4\pi^2}{3} \ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right) - \frac{3}{4} \left(\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right)\right)^3 \\
&- 6 \left(\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right)\right)^2 + 4 \left(\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{3}}\right)\right)^2 \ln\left(\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{3}}\right)\right) \\
&- \frac{\pi^2}{3} \ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right) + \frac{1}{6} \ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)^3 + \frac{3}{2} \left(\ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)\right)^2 \\
&- \left(\ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)\right)^2 \ln\left(\ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)\right) + \frac{2\pi^2}{3} \ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right) \\
&+ \frac{2}{3} \left(\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)\right)^3 - 6 \left(\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)\right)^2 \\
&+ 4 \left(\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)\right)^2 \ln\left(\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)\right) \\
&+ 4 \left(\ln\left(\frac{3}{2}\right)\right)^2 \sum_{n=1}^{\infty} \frac{(-)^n \zeta(2n)}{2n(2n+1)(2n+2)} \left(\frac{\ln\left(\frac{3}{2}\right)}{4\pi}\right)^{2n} \\
&- 16 \left(\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right)\right)^2 \sum_{n=1}^{\infty} \frac{(-)^n \zeta(2n)}{2n(2n+1)(2n+2)} \left(\frac{\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right)}{2\pi}\right)^{2n} \\
&- 16 \left(\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)\right)^2 \sum_{n=1}^{\infty} \frac{(-)^n \zeta(2n)}{2n(2n+1)(2n+2)} \left(\frac{\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)}{2\pi}\right)^{2n} \\
&+ 4 \left(\ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)\right)^2 \sum_{n=1}^{\infty} \frac{(-)^n \zeta(2n)}{2n(2n+1)(2n+2)} \left(\frac{\ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)}{2\pi}\right)^{2n},
\end{aligned} \tag{3.20}$$

where now four infinite series appear in the computation of  $Li_3\left(\frac{2}{3}\right)$ . The slowest convergence is found in the first infinite series with the argument  $x = \frac{\ln\left(\frac{3}{2}\right)}{4\pi} \approx \frac{1}{31}$ . The other three arguments are much smaller, at least by a factor of two. Inserting now in each of the four infinite series the polynomial representation of the even Bernoulli numbers (Eq. (3.1)) the combination of polylogarithmic identities for  $Li_3(x)$  with a polynomial description of Bernoulli numbers has been established, where the polynomial representation guarantees for an additional speed up in the convergence behavior of all of the four infinite series by more than an order of magnitude. Finally, at this approximation level  $\zeta(3)$  results to:



$$\begin{aligned}
\zeta(3) = & \frac{48\pi^2}{5} \ln\left(\frac{3}{2}\right) + \frac{6\pi^2}{5} \ln\left(\frac{4}{3}\right) - \frac{2\pi^2}{5} \ln\left(\frac{9}{8}\right) - \frac{96\pi^2}{5} \ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right) \\
& - \frac{48\pi^2}{5} \ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right) + \frac{28\pi^2}{5} \ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right) - \frac{108}{5} \left[\ln\left(\frac{3}{2}\right)\right]^2 \\
& - \frac{27}{5} \left[\ln\left(\frac{4}{3}\right)\right]^2 + \frac{9}{5} \left[\ln\left(\frac{9}{8}\right)\right]^2 + \frac{432}{5} \left[\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right)\right]^2 \\
& + \frac{432}{5} \left[\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)\right]^2 - \frac{108}{5} \left[\ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)\right]^2 - \frac{3}{5} \left[\ln\left(\frac{4}{3}\right)\right]^3 \\
& - \frac{6}{5} \left[\ln\left(\frac{3}{2}\right)\right]^3 + \frac{1}{5} \left[\ln\left(\frac{9}{8}\right)\right]^3 + \frac{54}{5} \left[\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right)\right]^3 \\
& - \frac{48}{5} \left[\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)\right]^3 - \frac{12}{5} \left[\ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)\right]^3 \\
& + \frac{18}{5} \left[\ln\left(\frac{4}{3}\right)\right]^2 \ln\left[\ln\left(\frac{4}{3}\right)\right] + \frac{72}{5} \left[\ln\left(\frac{3}{2}\right)\right]^2 \ln\left[\ln\left(\frac{3}{2}\right)\right] \\
& - \frac{6}{5} \left[\ln\left(\frac{9}{8}\right)\right]^2 \ln\left[\ln\left(\frac{9}{8}\right)\right] - \frac{288}{5} \left[\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right)\right]^2 \ln\left[\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right)\right] \\
& - \frac{288}{5} \left[\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)\right]^2 \ln\left[\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)\right] \\
& + \frac{72}{5} \left[\ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)\right]^2 \ln\left[\ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)\right] + \frac{864}{5} \sum_{n=1}^{\infty} c_n^{(1)} \left[\frac{\ln\left(\frac{4}{3}\right)}{2\sqrt{6}}\right]^{2n} \\
& - \frac{864}{5} \sum_{n=1}^{\infty} c_n^{(2)} \left[\frac{\ln\left(\frac{9}{8}\right)}{2\sqrt{6}}\right]^{2n} + \frac{13824}{5} \sum_{n=1}^{\infty} c_n^{(3)} \left[\frac{\ln\left(\frac{3}{2}\right)}{4\sqrt{6}}\right]^{2n} \\
& - \frac{13824}{5} \sum_{n=1}^{\infty} c_n^{(4)} \left[\frac{\ln\left(\frac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\right)}{2\sqrt{6}}\right]^{2n} - \frac{13824}{5} \sum_{n=1}^{\infty} c_n^{(5)} \left[\frac{\ln\left(\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}\right)}{2\sqrt{6}}\right]^{2n} \\
& + \frac{3456}{5} \sum_{n=1}^{\infty} c_n^{(6)} \left[\frac{\ln\left(\frac{5+2\sqrt{6}}{4\sqrt{6}}\right)}{2\sqrt{6}}\right]^{2n} .
\end{aligned} \tag{3.21}$$

Summing up again the first four terms ( $n=1,2,3,4$ ) from each of the six infinite series together with all terms given explicitly  $\zeta(3)$  follows now with an error of  $\delta \approx 0.37 * 10^{-21}$ . This is two orders of magnitude faster in the convergence when compared to the BBP formula [14].

It should be mentioned at this stage, that a further advantage of our series representation is that all of these six types of coefficients can be expressed in terms of elementary functions based on logarithmic expressions. This allows for a more detailed insight on  $\zeta(3)$  as it guarantees on

each approximation level an analytical expression for  $\zeta(3)$ .

The slowest convergence is now with  $Li_3(\frac{3}{4})$ . Therefore, we rewrite the polylogarithmic function belonging to the argument  $\frac{\ln(\frac{4}{3})}{2\sqrt{6}}$ , again with the help of the functional equation (3.10). It follows then:

$$\begin{aligned} Li_3\left(\frac{3}{4}\right) &= 8Li_3\left(\sqrt{\frac{3}{4}}\right) - 8Li_3\left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right) - 8Li_3\left(\frac{4}{2+\sqrt{3}}\right) \\ &+ 2Li_3\left(\frac{7+4\sqrt{3}}{8\sqrt{3}}\right) + 7\zeta(3) - \frac{2\pi^2}{3}\ln\left(\frac{4}{2+\sqrt{3}}\right) + \frac{4}{3}\left[\ln\left(\frac{4}{2+\sqrt{3}}\right)\right]^3, \end{aligned} \quad (3.22)$$

and finally:

$$\begin{aligned} Li_3\left(\frac{3}{4}\right) &= \zeta(3) - \frac{2\pi^2}{3}\ln\left(\frac{4}{3}\right) - \frac{1}{12}\left(\ln\left(\frac{4}{3}\right)\right)^3 + \frac{3}{2}\left(\ln\left(\frac{4}{3}\right)\right)^2 - \left(\ln\left(\frac{4}{3}\right)\right)^2 \ln\left(\ln\left(\frac{4}{3}\right)\right) \\ &+ \frac{4\pi^2}{3}\ln\left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right) + \frac{2}{3}\left(\ln\left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)\right)^3 - 6\left(\ln\left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)\right)^2 \\ &+ 4\left(\ln\left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)\right)^2 \ln\left(\ln\left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)\right) + 2\pi^2\ln\left(\frac{4}{2+\sqrt{3}}\right) + \frac{2}{3}\ln\left(\frac{4}{2+\sqrt{3}}\right)^3 \\ &- 6\ln\left(\frac{4}{2+\sqrt{3}}\right)^2 + 4\left(\ln\left(\frac{4}{2+\sqrt{3}}\right)\right)^2 \ln\left(\ln\left(\frac{4}{2+\sqrt{3}}\right)\right) - \frac{\pi^2}{3}\ln\left(\frac{7+4\sqrt{3}}{8\sqrt{3}}\right) \\ &- \frac{1}{6}\left(\ln\left(\frac{7+4\sqrt{3}}{8\sqrt{3}}\right)\right)^3 + \frac{3}{2}\left(\ln\left(\frac{7+4\sqrt{3}}{8\sqrt{3}}\right)\right)^2 \\ &- \left(\ln\left(\frac{7+4\sqrt{3}}{8\sqrt{3}}\right)\right)^2 \ln\left(\ln\left(\frac{7+4\sqrt{3}}{8\sqrt{3}}\right)\right) \\ &+ 4\left(\ln\left(\frac{4}{3}\right)\right)^2 \sum_{n=1}^{\infty} \frac{(-)^n \zeta(2n)}{2n(2n+1)(2n+2)} \left(\frac{\ln\left(\frac{4}{3}\right)}{4\pi}\right)^{2n} \\ &- 16\left(\ln\left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)\right)^2 \sum_{n=1}^{\infty} \frac{(-)^n \zeta(2n)}{2n(2n+1)(2n+2)} \left(\frac{\ln\left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)}{2\pi}\right)^{2n} \\ &- 16\left(\ln\left(\frac{4}{2+\sqrt{3}}\right)\right)^2 \sum_{n=1}^{\infty} \frac{(-)^n \zeta(2n)}{2n(2n+1)(2n+2)} \left(\frac{\ln\left(\frac{4}{2+\sqrt{3}}\right)}{2\pi}\right)^{2n} \\ &+ 4\left(\ln\left(\frac{7+4\sqrt{3}}{8\sqrt{3}}\right)\right)^2 \sum_{n=1}^{\infty} \frac{(-)^n \zeta(2n)}{2n(2n+1)(2n+2)} \left(\frac{\ln\left(\frac{7+4\sqrt{3}}{8\sqrt{3}}\right)}{2\pi}\right)^{2n} \end{aligned} \quad (3.23)$$

This procedure can be applied as often as necessary to compute  $\zeta(3)$  with a default accuracy. The only shortcoming is that the number of infinite series increases caused by the mathematical

structure of the functional equation (3.10), where the corresponding arguments appear as nested roots. As mentioned before, this procedure is not applicable to higher zeta values, as for example  $\zeta(5)$ , because no appropriate functional equations exist. In order to finally challenge the BBP formula [14] the polylogarithms  $Li_3\left(\sqrt{\frac{2}{3}}\right)$  and  $Li_3\left(\sqrt{\frac{3}{4}}\right)$  appearing with the slowest convergence behavior at this approximation level will be rewritten with the help of (3.10). It follows:

$$\begin{aligned}
Li_3\left(\sqrt{\frac{2}{3}}\right) &= 8Li_3\left(\sqrt[4]{\frac{3}{4}}\right) - 8Li_3\left(\frac{\sqrt[4]{2} + \sqrt[4]{3}}{2\sqrt[4]{2}}\right) - 8Li_3\left(\frac{2\sqrt[4]{3}}{\sqrt[4]{2} + \sqrt[4]{3}}\right) \\
&+ 2Li_3\left(\frac{\sqrt{2} + \sqrt{3} + 2\sqrt[4]{6}}{4\sqrt[4]{6}}\right) + 7\zeta(3) - \frac{2\pi^2}{3}\ln\left(\frac{2\sqrt[4]{3}}{\sqrt[4]{2} + \sqrt[4]{3}}\right) \\
&+ \frac{4}{3}\left[\ln\left(\frac{2\sqrt[4]{3}}{\sqrt[4]{2} + \sqrt[4]{3}}\right)\right]^3,
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
Li_3\left(\sqrt{\frac{3}{4}}\right) &= 8Li_3\left(\sqrt[4]{\frac{3}{4}}\right) - 8Li_3\left(\frac{\sqrt{2} + \sqrt[4]{3}}{2\sqrt[4]{3}}\right) - 8Li_3\left(\frac{2\sqrt{2}}{\sqrt{2} + \sqrt[4]{3}}\right) \\
&+ 2Li_3\left(\frac{2 + \sqrt{3} + 2\sqrt[4]{12}}{4\sqrt[4]{12}}\right) + 7\zeta(3) - \frac{2\pi^2}{3}\ln\left(\frac{2\sqrt{2}}{\sqrt{2} + \sqrt[4]{3}}\right) \\
&+ \frac{4}{3}\left[\ln\left(\frac{2\sqrt{2}}{\sqrt{2} + \sqrt[4]{3}}\right)\right]^3.
\end{aligned} \tag{3.25}$$

To further increase the speed up in the convergence behavior we use the polynom representation of the Bernoulli numbers where both  $B_{2n}$  and  $B_{2n-2}$  are involved [13]. It follows then for  $Li_3(x)$ :

$$\begin{aligned}
Li_3(x) &= \zeta(3) - \frac{\pi^2}{6}\ln\left(\frac{1}{x}\right) + \frac{1}{12}\left(\ln\left(\frac{1}{x}\right)\right)^3 + \frac{3}{4}\left(\ln\left(\frac{1}{x}\right)\right)^2 - \frac{1}{2}\left(\ln\left(\frac{1}{x}\right)\right)^2 \ln\ln\left(\frac{1}{x}\right) \\
&- \frac{1}{288}\left(\ln\left(\frac{1}{x}\right)\right)^4 + 24\left(\ln\left(\frac{1}{x}\right)\right)^2 \sum_{n=1}^{\infty} \left[ \frac{(-)^{n+1} \sum_{l=1}^{n+1} (-)^{l+1} \binom{n+5-l}{4} P^{(l)}(n+1)}{(2n-1)2n(2n+1)(2n+2)(2n+3)(2n+4)} \right. \\
&\left. - \sum_{n=1}^{\infty} \frac{2(-)^{n+1} \sum_{l=1}^n (-)^{l+1} \binom{n+2-l}{2} P^{(l)}(n)}{(2n-1)2n(2n+1)(2n+2)(2n+3)(2n+4)} \right] \left(\frac{\ln\left(\frac{1}{x}\right)}{2\sqrt{6}}\right)^{2n+2}
\end{aligned} \tag{3.26}$$

Calculating all relevant polylogarithms with the formula presented above the additional speed up in the convergence is more than one order of magnitude. Computing  $\zeta(3)$  at this approximation level, again by respecting the first four terms in the corresponding series ( $n=1,2,3,4$ ), the accuracy is better than  $10^{-25}$ . This is more than six orders of magnitude better in the convergence when compared to [14]. The complete numerical comparison with the BBP formula [14] for  $n=1,2,3$  and 4 is presented in Tab. I.

$\zeta(3)$	$\zeta(3)$ -Zeta series	$\zeta(3)$ -(Zeta series+BP+PL)	$\zeta(3)$ -BBP formula [14]
1st order n=1	$\delta=0.2*10^{-04}$	$\delta=0.1*10^{-10}$	$\delta=0.7*10^{-07}$
2nd order n=2	$\delta=0.2*10^{-06}$	$\delta=0.15*10^{-15}$	$\delta=0.4*10^{-11}$
3rd order n=3	$\delta=0.3*10^{-08}$	$\delta=0.2*10^{-20}$	$\delta=0.3*10^{-15}$
4th order n=4	$\delta=0.4*10^{-10}$	$\delta=0.2*10^{-25}$	$\delta=0.4*10^{-19}$

Table 1: Approximate computation of  $\zeta(3)$  as a function of the summation index  $n$  by use of the Clausen-function representation without and with use of the polynomial representation in combination with corresponding polylogarithmic identities. The numerical errors are compared to the BBP-type formula [14]

## 4 SUMMARY

In summary, we have presented a unique computational scheme for the explicit calculation of  $\zeta(3)$  by introducing a new series representation of  $\zeta(3)$ , which is based on the Clausen representation of odd integer zeta values. By an appropriate combination of polylogarithmic identities with a polynomial description of the even Bernoulli numbers, we were able to speed up the convergence behavior of the Clausen-based representation of  $\zeta(3)$  to a certain level which is significantly faster than that of the best BBP-type formulas available for  $\zeta(3)$ . Furthermore, we have presented a corresponding numerical comparison between our series representation and one of the best available BBP formulas. Furthermore, we have demonstrated using the first four terms in our series representation only that  $\zeta(3)$  can be computed with an accuracy of  $2 * 10^{-26}$ , where our computation guarantees on each approximation level for an completely analytical expression for  $\zeta(3)$ . Finally, we have shown that a computation by use of the combined polynomial representation of  $B_{2n}$  and  $B_{2n-2}$  further improves the approximate calculation of  $\zeta(3)$  by more than two orders of magnitude at all approximation levels.

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